Quasi-linear Utility and Two-Market Monopoly

By

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ABSTRACT

Given the ubiquitous assumption of quasi-linear utility in economic policy articles, this paper presents an overdue clarification of the implications of quasi-linear utility for two-market monopoly. The paper begins by deriving the demands facing a two-market monopoly from a representative consumer and then derives expressions for the profit margins expressed solely in terms of the own and cross-price elasticities of demand. The paper also analyzes the implications of quasi-linear utility for other issues in two-market monopoly: pricing below marginal cost in a market, third-degree price discrimination when the monopoly products are substitutes and pricing in the inelastic region of demands.

KEY WORDS

Quasi-linear Utility, Multiproduct Monopoly, Third-Degree Price Discrimination

JEL CODES

D42

1. INTRODUCTION

Thanks in large part to Varian (1985, 1992) the assumption of quasi-linear utility is ubiquitous in economic policy analyses because it allows one to simply measure social welfare as profit plus consumer surplus. This paper presents an overdue clarification of the implications of quasi-linear utility for the behavior of two-market monopoly and in the process corrects an error in Varian (1989, 1992). After deriving the demands facing a two-market monopoly from a representative consumer, the paper derives simple expressions for the monopoly profit margins expressed solely in terms of own and cross-price elasticities of demand. The paper also discusses three other implications of quasi-linear utility for two-market monopoly: (1) the possibility of pricing below marginal cost (2) the theory of third-degree price discrimination when the monopoly products are substitutes and (3) pricing in the inelastic region of demands.

2. QUASI-LINEAR UTILITY, INVERSE DEMANDS, DEMANDS AND ELASTICITIES

Let the utility function of the representative consumer be given by $u(x_0, x_1, x_2) = x_0 + \varphi(x_1, x_2)$, where x_0 is a numeraire good and x_1 and x_2 are the two goods produced by the monopolist.[1] Assume that φ is a twice continuously differentiable, strictly concave function where $\varphi_1 \equiv \frac{\partial \varphi}{\partial x_1} > 0, \varphi_2 \equiv \frac{\partial \varphi}{\partial x_2} > 0$, $\varphi_{11} \equiv \frac{\partial \varphi_1}{\partial x_1} < 0, \varphi_{22} \equiv \frac{\partial \varphi_2}{\partial x_2} < 0$, $\varphi_{12} \equiv \frac{\partial \varphi_2}{\partial x_1}$ and $\varphi_{11}\varphi_{22} > \varphi_{12}^2$.

[1] The analysis can easily be extended to the case where a monopolist sells *n* goods. For the case where the monopolist sells *n* goods let the utility function be $u(x_0, x_1, \cdots x_n) = x_0 + \varphi(x_1, \cdots x_n)$.

Let the price of the numeraire good be 1. Utility is maximized subject to the constraint: $x_0 + p_1 x_1 + p_2 x_2 \leq y$, where p_i is the price of good *i* and *y* is the consumer's exogenous income plus monopoly profit. Assuming an interior solution to the consumer's utility maximization problem, the consumer's inverse demand functions are: $p_1 = \varphi_1(x_1, x_2)$ and $p_2 = \varphi_2(x_1, x_2)$. Note that $\frac{\partial p_1}{\partial x_1} = \varphi_{11}, \frac{\partial p_2}{\partial x_2} = \varphi_{22}$ and, by Young's theorem, $\frac{\partial p_2}{\partial x_1} = \frac{\partial p_1}{\partial x_2} = \varphi_{12}$. Given the inverse demand functions, $\varphi_1(x_1, x_2)$ and $\varphi_2(x_1, x_2)$, one might ask under what conditions do the demand functions $x_1(p_1, p_2)$ and $x_2(p_1, p_2)$ exist and how are the partial derivatives of these demand functions related to the partial derivatives of the inverse demand functions?

Write the inverse demand functions as

(1) $F^i(p_i, x_1, x_2) = p_i - \varphi_i(x_1, x_2) = 0\,, \quad i = 1,2.$

Using the implicit function theorem we know that the demand functions $x_i = x_i(p_1, p_2)$, $i = 1,2$, exist in a neighborhood around a point that satisfies equation (1) as long as the Jacobian $|J| = \begin{vmatrix} \frac{\partial F^1}{\partial x_1} & \frac{\partial F^1}{\partial x_2} \\ \frac{\partial F^2}{\partial x_1} & \frac{\partial F^2}{\partial x_2} \end{vmatrix} = \varphi_{11}\varphi_{22} - \varphi_{12}^2 \neq 0$. Because $\varphi(x_1, x_2)$ is by assumption strictly concave it follows that the Jacobian is always positive and that the demand functions $x_i = x_i(p_1, p_2)$ do exist at every point satisfying equation (1).

The implicit function rule of differentiation applied to equation (1) yields:

(2) $\frac{\partial x_i}{\partial p_i} = \frac{\varphi_{jj}}{\varphi_{11}\varphi_{22} - \varphi_{12}^2} < 0 \quad i = 1,2; j = 1,2;\ i \neq j.$

(3) $\frac{\partial x_i}{\partial p_j} = \frac{-\varphi_{12}}{\varphi_{11}\varphi_{22} - \varphi_{12}^2}$ $\qquad i = 1,2; j = 1.2;\ i \neq j.$

It follows from equation (3) that $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$.[2] Furthermore, the sign of φ_{12} fully determines whether products 1 and 2 are substitutes, complements or independent. Products 1 and 2 are substitutes, that is $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} > 0$, if and only if φ_{12}<0.[3] Products 1 and 2 are complements, that is $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} < 0$, if and only if φ_{12}>0. The intuition for these latter results is that when products 1 and 2 are substitutes (complements) an increase in production of good 1 lowers (raises) the marginal utility of good 2 and lowers (raises) the maximum amount the representative consumer is willing to pay for good 2. If $\varphi_{12} = 0$, it follows that $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1} = 0$, the two demand functions are independent. Only in this special case will it be true that $\frac{\partial x_i}{\partial p_i} = \frac{1}{\varphi_{ii}} = \frac{1}{\frac{\partial p_i}{\partial x_i}}$.[4]

From the concavity of $\varphi(x_1, x_2)$ and equations (2) and (3) it also follows that $\frac{\partial x_1}{\partial p_1}\frac{\partial x_2}{\partial p_2} > \left(\frac{\partial x_1}{\partial p_2}\right)\left(\frac{\partial x_2}{\partial p_1}\right)$. Letting $e_{ij} \equiv \frac{\partial x_j}{\partial p_i}\frac{p_i}{x_j}$ be the elasticity of product *j* with respect to the price of product *i*, the previous inequality expressed in elasticity form is

(4) $e_{11}e_{22} - e_{12}e_{21} > 0$

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[2] Willig (1976) and Varian (1978) develop more general conditions for cross-price symmetry of the ordinary demand functions.

[3] Varian (1989, p. 619, 1992, p. 249) mistakenly states that $\frac{\partial p_2}{\partial x_1} > 0$ when products 1 and 2 are substitutes. Note also from equation (3) that $\frac{\partial x_i}{\partial p_j} \neq 1/\frac{\partial p_j}{\partial x_i}$ for $i \neq j$.

[4] Varian (1989, p. 618, 1992, p. 249) incorrectly assumes that $\frac{\partial x_i}{\partial p_i} = 1/\frac{\partial p_i}{\partial x_i}$ when the two products are substitutes.

3. PROFIT MAXIMIZATION FOR A TWO-PRODUCT FIRM

Let the profit function for a two-market monopoly be $\pi(p_1, p_2)\ (p_1 - c_1)x_1 + (p_2 - c_2)x_2$, where c_1 is constant marginal cost in market 1 and c_2 is constant marginal cost in market 2. The first order conditions for profit maximization are:

(5) $$\frac{\partial \pi}{\partial p_1} = x_1 + (p_1 - c_1)\frac{\partial x_1}{\partial p_1} + (p_2 - c_2)\frac{\partial x_2}{\partial p_1} = 0$$

(6) $$\frac{\partial \pi}{\partial p_2} = x_2 + (p_1 - c_1)\frac{\partial x_1}{\partial p_2} + (p_2 - c_2)\frac{\partial x_2}{\partial p_2} = 0$$

Let the profit margin in market *i* be $m_i = \frac{p_i - c_i}{p_i}$. Note that $m_i \leq 1$ and $m_i = 1$ for $c_i = 0$. Using the cross-price symmetry result that $\frac{\partial x_1}{\partial p_2} = \frac{\partial x_2}{\partial p_1}$, equations (5) and (6) with some algebraic manipulation may be rewritten as:

(7) $$m_1 e_{11} + m_2 e_{21} = -1$$

(8) $$m_1 e_{12} + m_2 e_{22} = -1$$

Solving for the profit maximizing values of the profit margins yields:

(9) $$m_1^* = \frac{-e_{22} + e_{21}}{e_{11}e_{22} - e_{12}e_{21}}$$

(10) $$m_2^* = \frac{-e_{11} + e_{12}}{e_{11}e_{22} - e_{12}e_{21}}$$

It is assumed that the second order sufficient conditions for a maximum are satisfied at the profit margins given by equations (9) and (10).[5] Note for $e_{12} = e_{21} = 0$, equations (9) and (10) simplify to $m_1^* = -\frac{1}{e_{11}}$ and $m_2^* = -\frac{1}{e_{22}}$.

The profit margin expressions given by equations (9) and (10) above are similar but somewhat simpler than expressions developed by Forbes (1988, 62). Forbes' expressions for the profit maximizing profit margins are: $m_1^* = \frac{-e_{22}+e_{12}\frac{R_2}{R_1}}{e_{11}e_{22}-e_{12}e_{21}}$ and $m_2^* = \frac{-e_{11}+e_{21}\frac{R_1}{R_2}}{e_{11}e_{22}-e_{12}e_{21}}$, where $R_i = p_i x_i$. Assuming cross-price symmetry it is easy to show that $e_{21} = e_{12}\frac{R_2}{R_1}$ and $e_{12} = e_{21}\frac{R_1}{R_2}$. Tirole's (1988,70) expressions for the profit maximizing profit margins in the two good case are: $m_1^* = -\frac{1}{e_{11}} - m_2^*\frac{R_2 e_{12}}{R_1 e_{11}}$ and $m_2^* = -\frac{1}{e_{22}} - m_1^*\frac{R_1 e_{21}}{R_2 e_{22}}$, which with some algebra can be shown to be equivalent to Forbes' expressions.

3.1 *Pricing Below Marginal Cost under Two-market Monopoly*

It has been well known for some time that if the two products are complements then it's possible that price may be below marginal cost in one market.[6] See Allen (1938, 359-62) and the references cited therein. Without loss of generality let $m_1^* > m_2^*$. From equation (10) the condition for price to be below marginal cost in market 2 in the present model is $e_{11} - e_{12} > 0$. Because e_{11} is assumed to be negative, for the profit margin to be negative in market 2 we must have e_{12} <0 and $|e_{12}| > |e_{11}|$. For example, a baseball team that sells beer (product 2) and

[5] It is interesting to note that when demands are linear the second order sufficient conditions for a maximum are:$e_{11} < 0$ and $e_{11}e_{22} > e_{12}e_{21}$. Thus the strict concavity of the utility function guarantees that the second order conditions for profit maximization are satisfied when demands are linear.

[6] If price were below marginal cost in both markets the monopolist's profit would be negative.

tickets (product 1) would price beer below its marginal cost if lowering the price of tickets by 1% led to a greater % increase in beer sold than the % increase in tickets sold.

3.2 *Third-Degree Price Discrimination when Demands are Interdependent*

Suppose we are dealing third-degree price discrimination where a monopolist sells a single product in two markets but the two markets are not perfectly sealed off from each other so that there is some substitution between the markets. See for example Varian's discussion (1989, 1992) of this subject. An example of this type of price discrimination is a movie theatre that shows afternoon movies at a lower price than the same movie shown at night. Because movies shown at night and during the day are substitutes, $e_{12}, e_{21} > 0$. Profit maximization for the price discriminating monopolist requires satisfying equations (9) and (10). The condition for $m_1^* > m_2^*$ is $-e_{22} + e_{21} > -e_{11} + e_{12}$. Note if $e_{12} = e_{21} = 0$ then the previous condition reduces to $|e_{22}| > |e_{11}|$, which is the well known result that a price discriminating monopoly with independent demands has a lower profit margin in the more elastic market.

3.3 *Pricing in the Inelastic Regions of Demands*

It is well known that an unregulated single-product monopolist will never operate in the price inelastic region of demand because that implies marginal revenue is negative. We now investigate whether this result extends to the case of a two-market monopoly. Because the profit margins in equations (9) and (10) must be less than or equal to 1 the following inequalities must hold:

(11) $e_{21}(e_{12} + 1) \leq e_{22}(e_{11} + 1)$ (holds with equality if $m_1^* = 1$)

(12) $e_{12}(e_{21} + 1) \leq e_{11}(e_{22} + 1)$ (holds with equality if $m_2^* = 1$)

If $|e_{11}| < 1$ then from inequality (11) we must have $e_{12} < 0, e_{21} < 0$ and $|e_{12}| < 1$. In words, if demand is inelastic in market 1, then the two products must be complements and the absolute value of e_{12} must be less than 1. Similarly, if demand is price inelastic in market 2 inequality (12) implies that the two products must be complements and that the absolute value of e_{21} must be less than 1. To verify that both markets can have price inelastic demands, let $e_{11} = -.8, e_{22} = -.9, e_{12} = e_{21} - .5$. From equations (9) and (10) we find $m_1^* = .851$ and $m_2^* = .638$.

If marginal cost is zero in a market, the two products are complements and the absolute values of the cross-price elasticities are less than 1, then it must be the case that demand is inelastic in that market. Consider the case where $c_1 = 0$ and hence $m_1^* = 1$. If $e_{12}, e_{21} < 0$, and $|e_{12}| < 1$ then it follows from equation (11) that $|e_{11}| < 1$. Similarly if $c_2 = 0$ and hence $m_2^* = 1$, then $e_{12}, e_{21} < 0$, and $|e_{21}| < 1$ implies $|e_{22}| < 1$. Finally, consider the case where $c_1 = c_2 = 0$ which implies $m_1^* = m_2^* = 1$, then $e_{12}, e_{21} < 0$, $|e_{12}| < 1$ and $|e_{21}| < 1$ implies $|e_{11}| < 1$ and $|e_{22}| < 1$.

Major league sports teams provide a good application of ticket pricing in the inelastic region of demand. All major league sports teams can be regarded as multiproduct monopolies selling complementary goods: tickets and concessions. The marginal cost of allowing another person to see a game is nearly zero as long as there are empty seats. It follows from the discussion above that ticket prices should be set in the inelastic region of demand as long as the absolute value of the cross-price elasticity of demand for concessions with respect to ticket prices is less than 1. Fort (2004) and Krautmann and Berri (2007) both argue forcefully that ticket pricing in the inelastic region of demand is commonplace in sporting events.

4. *CONCLUSION*

This paper has clarified several implications of the assumption of quasi-linear utility for two-market monopoly. After deriving the demands facing a two-market monopoly from a representative consumer, this paper derives expressions for profit maximizing profit margins expressed solely in terms of own and cross-price elasticities of demand. The paper also derives and discusses the condition for a two-market monopoly to price one of its products below marginal cost and clarifies the theory of third-degree price discrimination when the two monopoly products are substitutes. The paper ends with a discussion of pricing in the inelastic region of demand under two-market monopoly. It is shown that a two-market monopolist may (and in some conditions must) operate in the price inelastic region of demand in one or even both markets.

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