The Law of Diminishing Returns and the Generalized CES Production Function

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Abstract

There are two troubling and largely unrecognized implications of the generalized *CES* production function when there are increasing returns to scale and the elasticity of substitution (σ) exceeds one: (1) under these conditions the CES production function is always inconsistent with the law of diminishing marginal returns to either labor or capital and (2) the marginal product of labor (capital) always approaches infinity as labor (capital) approaches infinity. To avoid these two implications one must restrict the parameter values of the CES production function to either not allow for increasing returns to scale or to require that σ < 1.

Keywords: Generalized CES production function, increasing returns to scale, elasticity of substitution, diminishing marginal returns.

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I.   Introduction

George Stigler (1987, p. 129) referred to the law of diminishing returns discovered by

T.R. Malthus and Edward West in 1815 as "one of the heroic advances in the history of

economics." We can give a mathematical expression of this law as follows: Given a twice

differentiable production function $q = f(K, L)$ that expresses, for a given state of technology,

the relationship between output (*q)* and two inputs capital (*K*) and labor (*L*), then for any given

value of *K* we can find a value of *L* say $L_0(K)$ such that $f_{LL} < 0$ for $L > L_0(K)$ and for any given

value of *L* we can find a value of *K,* say $K_0(L)$ such that $f_{KK} < 0$ for $K > K_0(L)$.[1] Some of the

most commonly used production functions such as the constant returns to scale Cobb-Douglas

production function and the constant returns to scale CES production functions exhibit

diminishing marginal productivities everywhere - $f_{LL} < 0$ and $f_{KK} < 0$ for all values of *K* and *L*.

In economic growth theory, production functions are often assumed to be subject to

diminishing returns in order to guarantee a steady state equilibrium. Barro and Sala-i-Martin

(2004, p. 27) list diminishing returns to private inputs as one of the five characteristics of a

neoclassical production function. Additionally, in microeconomic theory the second order

sufficient conditions for a profit maximum for a price-taking firm require that $f_{LL} < 0$ and

$f_{KK} < 0$ at the optimal values of *K* and *L*.[2]

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[1] This is consistent with the law of diminishing returns expressed by Stigler (1987, p. 129) and Menger (1954, p.424).
[2] Consider a price-taking firm with a profit function $\pi(K, L) = pf(K, L) - wL - vK$, where *w* and *v* are constant input prices. The first order conditions are: $pf_L - w = 0$ and $pf_K - v = 0$. The second order sufficient conditions require that $pf_{LL} < 0$ and $pf_{KK} < 0$ at the optimal values of *K* and *L*. This powerful argument for diminishing returns, however, only applies to firms with production functions that exhibit decreasing returns to scale. Firms with increasing returns to scale production functions are not compatible with competitive price taking markets.

This paper discusses the law of diminishing marginal returns as well as the asymptotic behavior of average and marginal products in the generalized CES production function developed by Brown and de Cani (1963). The generalized CES production function allows for either decreasing, constant or increasing returns to scale and is a generalization of the original CES production function developed by Arrow, Chenery, Minhas and Solow (1961) which assumes constant returns to scale.[3] Because the generalized *CES* production function allows for the estimation of returns to scale as well as the elasticity of substitution it is a tempting choice for the estimation of industry and aggregate production functions.[4]

Given the attractiveness of the generalized CES production function it is important to be aware of all of its theoretical implications. There are two troubling and largely unrecognized implications of the generalized *CES* production function when there are increasing returns to scale and the elasticity of substitution (σ) exceeds one*: (1) in this case the generalized CES production function is always inconsistent with the law of diminishing marginal returns to either labor or capital* and (2) *both the average and marginal products of labor (capital) always approach infinity as L (K) approaches infinity.* To avoid these two implications one must restrict the parameter values of the CES production function to not simultaneously allow for increasing returns to scale and $\sigma > 1$.[5]

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[3] Ferguson (1969, 101) also credits Brown and de Cani (1963) and Kendrick and Sato (1963) with independently discovering the CES production function.

[4] For early examples of generalized CES production function estimates see Brown and de Cani (1963) and Ferguson (1965) and for a recent example see Shen and Whalley (2013). See Klump, McAdam and Willman (2012) for a good discussion of some of the problems and recent advances in the estimation of CES production functions.

[5] It is interesting to note that La Grandville (1989, p. 481) has shown that $\sigma > 1$ is a necessary condition for output per person to go to infinity in a constant returns to scale CES Solow growth model.

II.   THE GENERALIZED CES PRODUCTION FUNCTION

The generalized CES production function is given by

(1) $q = f(K,L) = A[\delta K^\rho + (1-\delta)L^\rho]^{\frac{\varepsilon}{\rho}}$ $A > 0, 0 < \delta < 1, \rho \le 1, \varepsilon > 0, \rho \ne 0.$

The generalized CES production function is homogeneous of degree ε. For $\varepsilon>1$ there are

increasing returns to scale, for $\varepsilon=1$ there are constant returns to scale and for $\varepsilon <1$ there are

decreasing returns to scale. The elasticity of substitution is $\sigma = \frac{1}{1-\rho}$. Note that for $\rho = 1$,

$\sigma = \infty$, for $\rho = 0, \sigma = 1$ and for $\rho = -\infty, \sigma = 0$. In the limit as ρ approaches 0, the generalized

CES production function approaches the Cobb-Douglas production function

(2) $q = AK^{\delta\varepsilon}L^{(1-\delta)\varepsilon}$ $\rho = 0.$

Because the Cobb-Douglas case is well understood, from this point onwards it will be

assumed that $\rho \ne 0$. The average products of labor and capital, respectively, are:

(3) $\frac{q}{L} = \frac{A[\delta K^\rho + (1-\delta)L^\rho]^{\frac{\varepsilon}{\rho}}}{L}$

and

(4) $\frac{q}{K} = \frac{A[\delta K^\rho + (1-\delta)L^\rho]^{\frac{\varepsilon}{\rho}}}{K}$.

The marginal products of labor and capital are given, respectively, by

$$(5)\ f_L = \frac{\partial q}{\partial L} = \varepsilon A[\delta K^\rho + (1-\delta)L^\rho]^{\frac{\varepsilon}{\rho}-1}(1-\delta)L^{\rho-1} = \varepsilon \frac{q}{L}\Omega_L$$

and

$$(6)\ f_K = \frac{\partial q}{\partial K} = \varepsilon A[\delta K^\rho + (1-\delta)L^\rho]^{\frac{\varepsilon}{\rho}-1}\delta K^{\rho-1} = \varepsilon \frac{q}{K}\Omega_K.$$

In equations (5) and (6) Ω_L and Ω_K are defined by

$$(7)\ \Omega_L = \frac{(1-\delta)L^\rho}{\delta K^\rho + (1-\delta)L^\rho} = \frac{1}{\left[\frac{\delta}{1-\delta}\left(\frac{K}{L}\right)^\varrho + 1\right]}$$

and

$$(8)\ \Omega_K = \frac{\delta K^\rho}{\delta K^\rho + (1-\delta)L^\rho} = \frac{1}{\left[\frac{(1-\delta)}{\delta}\left(\frac{L}{K}\right)^\varrho + 1\right]}.$$

Note from equations (7) and (8) $0 < \Omega_L < 1$ and $0 < \Omega_K < 1$. Also the marginal and average products of capital and labor for the generalized CES production function are positive for all values of *K* and *L*.

III.   ASYMPTOTIC PROPERTIES OF AVERAGE AND MARGINAL PRODUCTS FOR $\rho > 0$

This section of the paper derives the asymptotic behavior of average and marginal products for the interesting case where $\rho > 0$ and returns to scale are variable. [6]

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[6] Arrow, Chenery, Minhas and Solow (1961, p. 230-31) analyze the asymptotic properties of the average product of labor under the CES with constant returns to scale and Brown (1966, p. 50) and Ferguson (1969 pp. 104-

Equation (3) may be rewritten as

$$(9)\ \frac{q}{L} = A[\delta K^\rho + (1-\delta)L^\rho]^{\frac{\varepsilon}{\rho}}L^{-1} = A[L^\rho\{\delta\left(\frac{K}{L}\right)^\varrho + 1 - \delta\}]^{\frac{\varepsilon}{\rho}}L^{-1}$$

$$= AL^{\varepsilon-1}\{\delta\left(\frac{K}{L}\right)^\varrho + 1 - \delta\}^{\frac{\varepsilon}{\rho}}.$$

For $\rho > 0$, the expression in curly brackets above approaches $1 - \delta$ as L goes to infinity. Hence,

$\lim_{L\to\infty}\frac{q}{L} = \lim_{L\to\infty} AL^{\varepsilon-1}(1-\delta)^{\frac{\varepsilon}{\rho}}$. For ε <1 $\lim_{L\to\infty}\frac{q}{L} = 0$, for ε =1 $\lim_{L\to\infty}\frac{q}{L} = A(1-\delta)^{\frac{\varepsilon}{\rho}}$ and

for ε >1 $\lim_{L\to\infty}\frac{q}{L} = \infty$. Notice in this last case where $\rho > 0$ and ε >1 the average product of

labor approaches infinity as labor approaches infinity even when ε is only infinitesimally larger

than 1. A similar analysis for the average product of capital will show that for ε <1, $\lim_{K\to\infty}\frac{q}{K} =$

0, for ε =1, $\lim_{K\to\infty}\frac{q}{K} = A(\delta)^{\frac{\varepsilon}{\rho}}$ and for ε >1, $\lim_{K\to\infty}\frac{q}{K} = \infty$. Finally, regardless of the value of

ε, it is easy to see from equation (9) that $\lim_{K\to\infty}\frac{q}{L} = \infty$.

From equation (5) we have $f_L = \varepsilon\frac{q}{L}\Omega_L$, where Ω_L is defined by equation (7). For $\rho > 0$,

$\lim_{L\to\infty}\Omega_L = 1$. Hence, for ε <1 $\lim_{L\to\infty} f_L = 0$, for ε =1, $\lim_{L\to\infty} f_L = \varepsilon A(1-\delta)^{\frac{\varepsilon}{\rho}}$ and for ε >1,

$\lim_{L\to\infty} f_L = \infty$. A similar analysis for f_K will show that for ε <1, $\lim_{K\to\infty} f_K = 0$, for ε =1,

$\lim_{K\to\infty} f_K = \varepsilon A(\delta)^{\frac{\varepsilon}{\rho}}$ and for ε >1, $\lim_{K\to\infty} f_K = \infty$.

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105) analyze the asymptotic properties of marginal products under the CES with constant returns to scale. If $\rho < 0$ then it is easy to show using equations (3) and (4) that regardless of the returns to scale, the marginal and average products of the inputs approach 0 as the variable input approaches infinity.

Obviously in the case where $\rho > 0$ and $\varepsilon > 1$, where the average and marginal products of labor (capital) approaches infinity as labor (capital) approaches infinity, this is a violent violation of the law of diminishing marginal returns to labor and capital. The implications of this case are striking for economic growth theory. In this case output per unit of labor can rise without limit with labor force growth even if there is no technological progress and no capital accumulation. Also worth noting for $\rho > 0$, regardless of the value of ε, output per unit of labor rises without limit if there is continual capital accumulation with no population change and no change in technology.

IV. THE LAW OF DIMINISHING MARGINAL RETURNS

As mentioned in the introduction there is diminishing marginal returns to labor if $f_{LL} < 0$ for $L > L_0(K)$ for any given value of K. Differentiating equation (5) with respect to L gives

$$(10)\ f_{LL} = \left\{\varepsilon \frac{q}{L}\Omega_L L^{-2}\right\}\{(\varepsilon - \rho)\Omega_L + \rho - 1\}.$$

The first expression in curly brackets on the right hand side of equation (10) is always positive. Let the second expression in curly brackets on the right hand side of equation (10) be B_2. To determine the sign of B_2, note $0 < \Omega_L < 1$. It follows for $\rho < 1$ and $\varepsilon \leq 1$ that $B_2 < 0$ and $f_{LL} < 0$ for all values of L. Also, for < 0, $\lim_{L\to\infty} \Omega_L = 0$ and $\lim_{L\to\infty} B_2 = \rho - 1$. Hence in this case f_{LL} must turn negative for large enough values of L, regardless of the value of ε. For

the Cobb-Douglas case, $\rho = 0$, it is easy to verify from equation (2) $f_{LL} < 0$ for all values of L as long as $(1-\delta)\varepsilon < 1$.

For values of ε much larger than 1, the sign of B_2 can obviously be positive. This was recognized by Brown (1966, p. 47): "But we should expect in any reasonable neoclassical model that strong economies of scale would prevent a decline in the marginal product of labor. Hence, the possibility that the CES production function does not satisfy the second criterion for this special case is not disturbing." In a follow up footnote on the same page Brown goes on to add: "A similar situation was found above to exist for the Cobb-Douglas production function."

What Brown did not seem to realize was that when $\rho > 0$, for *any* value of ε greater than 1, f_{LL} must become positive for large enough values of L. In other words when there are increasing returns to scale and the elasticity of substitution exceeds one, the CES production function is always inconsistent with the law of diminishing marginal returns. To see this recall for $\rho > 0$, $\lim_{L\to\infty} \Omega_L = 1$. It follows for $\rho > 0$ that $\lim_{L\to\infty} B_2 = \varepsilon - 1$. Hence for $\varepsilon > 1$ and $\rho > 0$ f_{LL} must always become positive for large enough values of L.

While Brown (1966) may not have fully realized all the implications of the generalized CES production function he certainly did realize it could generate positive values of f_{LL}. This result has apparently been forgotten. For example, the current edition (eleven) of "Microeconomics: Basic Principles and Extensions," by Nicholson and Snyder (2012, p. 330), a widely used graduate textbook in Microeconomics, asserts that the many input generalized CES production function $q = \left[\sum \alpha_i x_i^\rho\right]^{\frac{\varepsilon}{\rho}}$ "exhibits diminishing marginal productivities for each input because $\rho \le 1$." Nearly identical statements occur in editions 4-10 of this popular textbook.

Clearly these statements are incorrect for $\varepsilon > 1$ and $\rho > 0$. Not only are there *not* diminishing

marginal productivities for this subcase but the marginal productivities approach infinity as the

relevant variable input approach infinity.[7]

Henderson and Quandt (1980, p. 68) refer to "the almost universal *law of diminishing*

marginal product." Yet in their discussion of the generalized CES production function they fail to

mention that this production function is inconsistent with the law of diminishing marginal

product. The more recent advanced graduate Microeconomic texts by Varian (1992), Mas-

Colell, Whinston, and Green (1995), and Jehle and Reny (2001) and Riley (2012) only discuss the

constant returns to scale CES production function.[8] Interestingly, none of these four texts

explicitly discuss the law of diminishing marginal returns but all four discuss diminishing rates of

technical substitution or quasi-concavity of the production function.

V.   DIMINISHING RATES OF TECHNICAL SUBSTITUTION

The rate of technical substitution (RTS) for the generalized CES is

$$(11)\ RTS = \frac{f_L}{f_K} = \frac{(1-\delta)}{\delta} \left(\frac{K}{L}\right)^{1-\rho}.$$

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[7] For the many input generalized CES production function $q = \left[\sum \alpha_i x_i^\rho\right]^{\frac{\varepsilon}{\rho}}$ in can be shown, using the same method used in section 3, for $\varepsilon > 1$ and $\rho > 0$ that the average and marginal products of each input i approaches infinity as x_i approaches infinity. Also it can be shown, using the same method used in section 4, for $\varepsilon > 1, \rho > 0$ and for large enough values of x_i, $\frac{\partial^2 q}{\partial x_i^2} > 0$.

[8] The mathematical economics texts by Simon and Blume (1994) and Silberberg and Suen (2001) also discuss only the constant returns to scale CES production function. Chiang and Wainwright (2005) discuss the constant returns to scale CES production and briefly mentions the generalized CES on p. 401.

It is worth noting that the expression for *RTS* in equation (11) does not even contain the returns to scale parameter ϵ. The generalized CES for $\rho < 1$ exhibits a diminishing rate of technical substitution and all its isoquants are strictly convex.[9] Thus, values of *K* and *L* that satisfy $RTS = \frac{w}{v}$ minimize the cost of producing any given level of output. This remains true whether there is diminishing marginal productivity or increasing marginal productivity.

The relationship between diminishing marginal factor returns and diminishing rates of technical substitution can be seen from examining the sufficient condition for strict quasi-concavity of the production function (which is equivalent to diminishing rates of technical substitution):

$$(12)\ f_{LL}f_k^2 - 2f_{KL}f_K f_L + f_{kk}f_L^2 < 0 \ \text{for all values of } K \text{ and } L.$$

The condition above is always satisfied for the generalized CES production function for $\rho < 1$ even when f_{LL} or f_{KK} are positive, as was shown to be possible in section IV. Thus if one wishes to insure that the law of diminishing returns holds in the generalized CES, one requires more than quasi-concavity of the production function. The two conditions that will insure diminishing returns in the generalized CES are either non-increasing returns to scale ($\varepsilon \leq 1$) or an elasticity of substitution less than one ($\sigma < 1$).

VI.   CONCLUSION

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[9] From equation (11) $\frac{dRTS}{dL} = -\frac{(1-\rho)}{L}\frac{(1-\delta)}{\delta}\left(\frac{K}{L}\right)^{-\rho}\left[\frac{(1-\delta)}{\delta}\left(\frac{K}{L}\right)^{1-\rho} + \frac{K}{L}\right] < 0$ for all *L* if $\rho < 1$.

This paper has shown that the law of diminishing marginal returns is not always satisfied in the generalized CES production function. To insure that this law holds in the generalized CES one must require either non-increasing returns to scale ($\varepsilon \leq 1$) or an elasticity of substitution less than one ($\sigma < 1$). Allowing increasing returns to scale ($\varepsilon > 1$) and an elasticity of substitution greater than one ($\sigma > 1$) is not only inconsistent with the law of diminishing returns but it implies that the average and marginal products of the variable inputs approach infinity as the variable input approaches infinity.

REFERENCES

Arrow, K. J., Chenery, H. B., Minhas, B. S. and Solow, R. M. (1961) Capital-labor substitution and economic efficiency. *Review of Economics and Statistics* 43(3): 225-250.

Barro, R. J. and Sala i-Martin, X. (2004) *Economic Growth.* 2nd edn. Cambridge, MA: MIT Press.

Brown, M. (1966) *On the Theory and Measurement of Technological Change.* Cambridge University Press.

Brown, M. and de Cani, J. S. (1963) Technological change and the distribution of income. *International Economic Review, 4(3): 289-309.*

Chiang, A. C. and Wainwright, K. (2005) *Fundamental Methods of Mathematical Economics.* Fourth edn. McGraw-Hill Irwin.

Ferguson, C. E. (1965) Substitution, technical progress, and returns to scale. *American Economic Review* 55(1/2): 296-305.

Ferguson, C. E. (1969) *The Neoclassical Theory of Production and Distribution.* Cambridge University Press.

Henderson, J. M. and Quandt, R. E. (1980) *Microeconomic Theory: A Mathematical Approach.* Third edn. New York: McGraw-Hill Book Company.

Jehle, G. A. and Reny, P. J. (2011) *Advanced Microeconomic Theory. Third edn.* Financial Times Prentice Hall.

Kendrick, J. W. and Sato, R. (1963) Factory prices, productivity and growth. *American Economic Review*, 53(5) 974-1003.

La Grandville, Oliver de (1989) In quest of the Slutsky diamond, *American Economic Review, 79(3): 468-481.*

Mas-Collel, A., Whinston., M. D. and Green, J.R. (1995) *Microeconomic Theory.* Oxford University Press.

Menger, K. (1954) The laws of return, in O. Morgenstern (ed.), *Economic Activity Analysis.* New York: Wiley, 419-482.

Nicholson, W. and Snyder, C. M. (2012) *Microeconomic Theory: Basic Principles and Extentions.* Eleventh edn. Mason, OH: South-Western.

Sato, R. (1987) CES production function, in J. Eatwell *et al.* (eds.), *The New Palgrave A Dictionary of Economics,* Vol. 1, London: Macmilian, 395-396.

Shen, K. and Whalley, J. (2013) Capital-labor substitutions in nested CES production functions for China. NBER working paper 19104. National Bureau of Economic Research.

Silberberg E. and Suen, W. (2001) *The Structure of Economics: A Mathematical Analysis. Third edn.* Irwin McGraw Hill.

Simon, C. P. and Blume L. (1994) *Mathematics for Economists* W. W. Norton & Company.

Solow, R. M. (1956) A contribution to the theory of growth. *Quarterly Journal of Economics, 70(1): 65-94.*

Stigler, G. J. (1987) *The Theory of Price.* Fourth edn. New York: Macmillion Publishing Company.

Varian H. R. (1992) *Microeconomic Analysis. Third edn.* W.W. Norton & Company.

Riley, G. R. (2012) *Essential Microeconomics.* Cambridge University Press.